AP® Calculus AB **Lesson Plan**

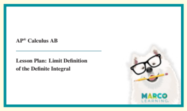
Limit Definition of the Definite Integral

Duration

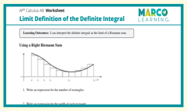
One 90-minute class period

Resources

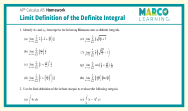
1. Presentation



2. Worksheet



3. Homework



Objectives of Lesson

- To interpret a definite integral as the limit of a Riemann Sum
- To be able to represent a definite integral as the limit of a Riemann Sum
- To be able to evaluate a definite integral using properties of limits and summations

College Board Objectives from the 2019–20 *CED*

- **Mathematical Practices—Practice 1:** Implementing Mathematical Processes
- **Mathematical Practices—Practice 1F:** Determine expressions and values using mathematical procedures and rules.
- **Mathematical Practices—Practice 2:** Connecting Representations
- **Mathematical Practices—Practice 2C:** Identify a re-expression of mathematical information
- presented in a given representation.
- **Mathematical Practices—Practice 4:** Communication and Notation—Use correct notation, language, and mathematical conventions to communicate results or solutions.
- **Mathematical Practices—Practice 4C:** Use appropriate mathematical symbols and notation.
- **Learning Objective LIM-5.B:** Interpret the limiting case of the Riemann Sum as a definite integral.
- **Learning Objective LIM-5.C:** Represent the limiting case of the Riemann Sum as a definite integral.
- **Prior Knowledge:** Students should be able to compute the value of a left-, right-, and midpoint Riemann Sum from work in previous lessons.

NOTES

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How to Use This Lesson Plan

The slide presentation is meant to be used during a whole group discussion. Students fill in the information on the worksheet as the lesson progresses. Following the presentation, debrief with students and assign practice exercises for homework.

Instructions

1. Whole Group Discussion (slides 1–4). Slides 1–2 are meant to help develop the definition of the definite integral. The first example shows a right-hand Riemann Sum. Discuss the meaning of the notation depicted in the graph, and emphasize that, although there appear to be 10 rectangles, the graph is meant to be generalized to *n* rectangles (subintervals). Students may struggle to understand the meaning of *k* in this context, so be sure to discuss the meaning of *x_k* as the *k*th rectangle (subinterval) and identify *k* as a "counter."
2. Formative Assessment (slides 5–6). Check for understanding using voting cards or think-pair-share.
3. Whole Group Discussion (slides 7–12). Work through the examples on slides 9 and 11 together, emphasizing the importance of proper mathematical notation as you go. Students are not required to memorize the summation properties and formulas. They should, however, highlight these properties and formulas in their notes, so they can refer to them when working through the homework exercises.
4. Formative Assessment (slide 13). Have students work independently to solve the problem on slide 13.
5. Follow-Up Questions. Debrief with a whole-group discussion:
   a. How are Riemann Sums used to approximate the area under a curve on a given interval?
   b. How are limits used to find the exact area under a curve on a given interval?
   c. The examples shown involved using a right-hand Riemann Sum. Would this technique work for a left-hand Riemann Sum? A midpoint Riemann Sum?

NOTES

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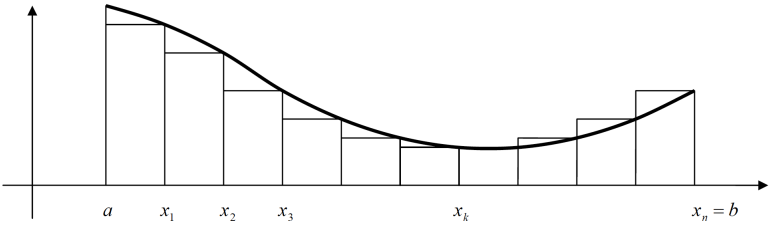
AP® Calculus AB

Lesson Plan: Limit Definition of the Definite Integral

Limit of a Riemann Sum

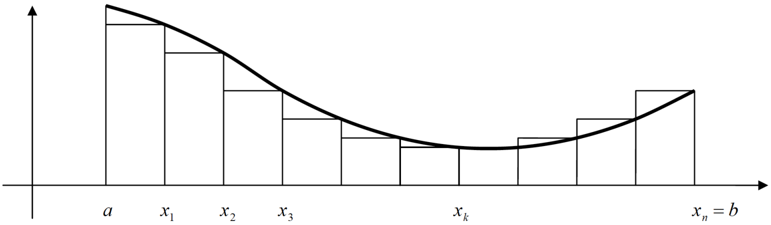
Write an expression for each of the following quantities:



1. the number of rectangles.
2. the width of each rectangle.
3. the height of each rectangle.
4. the area of each rectangle.
5. the total area of all rectangles.

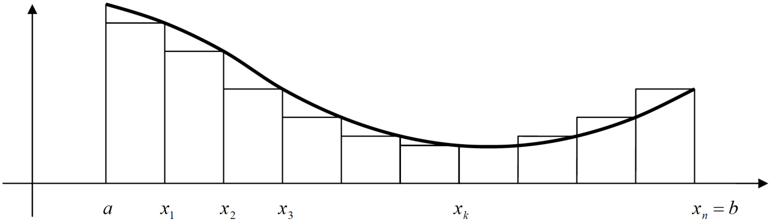
Limit of a Riemann Sum

Write an expression for each of the following quantities:



1. the number of rectangles. n
2. the width of each rectangle. $\Delta x = \frac{b-a}{n}$
3. the height of each rectangle. $f(x_k)$
4. the area of each rectangle. $f(x_k)\Delta x$
5. the total area of all rectangles.
   $\sum_{k=1}^{n} f(x_k)\Delta x$

Limit of a Riemann Sum



$$x_1 = a + \Delta x$$

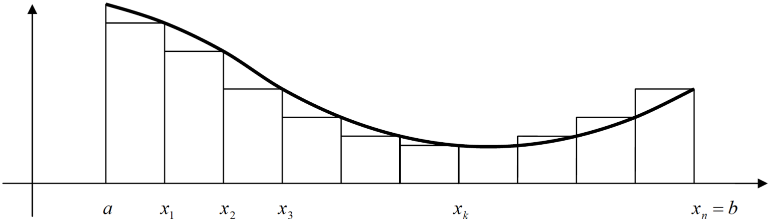
$$x_2 =$$

$$x_3 =$$

$$\vdots$$

$$x_k =$$

Limit of a Riemann Sum



$$x_1 = a + \Delta x$$

$$x_2 = a + 2\Delta x$$

$$x_3 = a + 3\Delta x$$

$$\vdots$$

$$x_k = a + k\Delta x$$

Definition of Definite Integral

If f is a continuous function defined on $[a, b]$, and if:

- $[a, b]$ is divided into n equal subintervals of width $\Delta x = \frac{b-a}{n}$,
- and if $x_k = a + k\Delta x$ is the right endpoint of subinterval k,

then the definite integral of f from a to b is the number

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\, \Delta x$$

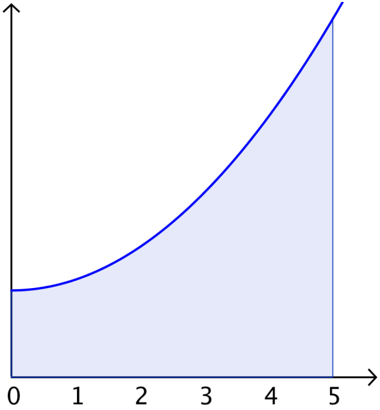
Let n be the number of subintervals. The exact area under the curve is given by the limit:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{5}{n}\right)\left(\left(\frac{5k}{n}\right)^2 + 2\right)$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\left(\frac{5k}{n}\right)^2 + 2\right) \Delta x$$

which is exactly equal to:

$$\int_0^5 (x^2 + 2)\, dx$$



Definite Integral as the Limit of a Riemann Sum

Which of the following limits is equal to $\int_3^5 x^4\, dx$?

(A) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3 + \frac{k}{n}\right)^4 \frac{1}{n}$

(B) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3 + \frac{k}{n}\right)^4 \frac{2}{n}$

(C) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3 + \frac{2k}{n}\right)^4 \frac{1}{n}$

(D) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3 + \frac{2k}{n}\right)^4 \frac{2}{n}$

Which of the following limits is equal to $\int_3^5 x^4\, dx$?

(A) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3 + \frac{k}{n}\right)^4 \frac{1}{n}$

(B) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3 + \frac{k}{n}\right)^4 \frac{2}{n}$

(C) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3 + \frac{2k}{n}\right)^4 \frac{1}{n}$

(D) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3 + \frac{2k}{n}\right)^4 \frac{2}{n}$

Which of the following limits is equal to $\int_2^5 (4-2x)\,dx$?

(A) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(4-2\left(2+\frac{k}{n}\right)\right)\frac{1}{n}$

(B) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(4-2\left(2+\frac{3k}{n}\right)\right)\frac{1}{n}$

(C) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(4-2\left(2+\frac{k}{n}\right)\right)\frac{3}{n}$

(D) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(4-2\left(2+\frac{3k}{n}\right)\right)\frac{3}{n}$

Which of the following limits is equal to $\int_2^5 (4-2x)\,dx$?

(A) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(4 - 2\left(2 + \frac{k}{n}\right)\right) \frac{1}{n}$

(B) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(4 - 2\left(2 + \frac{3k}{n}\right)\right) \frac{1}{n}$

(C) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(4 - 2\left(2 + \frac{k}{n}\right)\right) \frac{3}{n}$

(D) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(4 - 2\left(2 + \frac{3k}{n}\right)\right)^4 \frac{3}{n}$

Summation Properties

Recall the following summation properties:

1. $\sum_{k=1}^{n} c =$

2. $\sum_{k=1}^{n} ca_k =$

3. $\sum_{k=1}^{n} (a_k \pm b_k) =$

Summation Properties

Recall the following summation properties:

1. $\sum_{k=1}^{n} c = nc$

2. $\sum_{k=1}^{n} ca_k = c \sum_{k=1}^{n} a_k$

3. $\sum_{k=1}^{n} (a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$

Summation Formulas

Recall the following summation formulas:

1. $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

2. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

3. $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

Use the definition of definite integral to evaluate $\int_3^5 (3x + 1)\,dx$.

First, we write an expression for Δx.

$$\Delta x =$$

Second, we write an expression for x_k.

$$x_k =$$

Third, we write the limit definition of the definite integral.

$$\int_3^5 (3x + 1)\,dx =$$

Use the definition of definite integral to evaluate $\int_3^5 (3x+1)\,dx$.

First, we write an expression for Δx.

$$\Delta x = \frac{2}{n}$$

Second, we write an expression for x_k.

$$x_k = 3 + \frac{2k}{n}$$

Third, we write the limit definition of the definite integral.

$$\int_3^5 (3x+1)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} [3(x_k)+1]\,\Delta x$$

Limit Definition of Definite Integral

$$\begin{aligned}
\int_3^5 (3x+1)\,dx &= \lim_{n\to\infty} \sum_{k=1}^{n} [3(x_k)+1]\,\Delta x \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left[3\left(3+\frac{2k}{n}\right)+1\right]\frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{20}{n}+\frac{12k}{n^2}\right) \\
&= \lim_{n\to\infty} \left[\frac{20}{n}\sum_{k=1}^{n} 1+\frac{12}{n^2}\sum_{k=1}^{n} k\right] \\
&= \lim_{n\to\infty} \left[\frac{20}{n}\cdot n+\frac{12}{n^2}\cdot\frac{n(n+1)}{2}\right] \\
&= 20+6 \\
&= 26
\end{aligned}$$

Use the definition of definite integral to evaluate $\int_0^4 (2x^2 + 3)\,dx$.

First, we write an expression for Δx.

$$\Delta x =$$

Second, we write an expression for x_k.

$$x_k =$$

Third, we write the limit definition of the definite integral.

$$\int_0^4 (2x^2 + 3)\,dx =$$

Limit Definition of Definite Integral

Use the definition of definite integral to evaluate $\int_0^4 (2x^2 + 3)\,dx$.

First, we write an expression for Δx.

$$\Delta x = \frac{4}{n}$$

Second, we write an expression for x_k.

$$x_k = \frac{4k}{n}$$

Third, we write the limit definition of the definite integral.

$$\int_0^4 (2x^2 + 3)\,dx = \lim_{n \to \infty} \sum_{k=1}^{n} [2(x_k)^2 + 3]\,\Delta x$$

Limit Definition of Definite Integral

$$\begin{aligned}
\int_0^4 (2x^2+3)\,dx &= \lim_{n\to\infty}\sum_{k=1}^{n}[2(x_k)^2+3]\,\Delta x \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\left[2\left(\frac{4k}{n}\right)^2+3\right]\frac{4}{n} \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{128}{n^3}k^2+\frac{12}{n}\right) \\
&= \lim_{n\to\infty}\left[\frac{128}{n^3}\sum_{k=1}^{n}k^2+\frac{12}{n}\sum_{k=1}^{n}1\right] \\
&= \lim_{n\to\infty}\left[\frac{128}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{12}{n}\cdot n\right] \\
&= \frac{128}{3}+12=\frac{164}{3}
\end{aligned}$$

Use the definition of definite integral to evaluate $\int_2^5 (8x - x^2)\, dx$.

Limit Definition of Definite Integral

Use the definition of definite integral to evaluate $\int_2^5 (8x - x^2)\,dx$. 45

Credits

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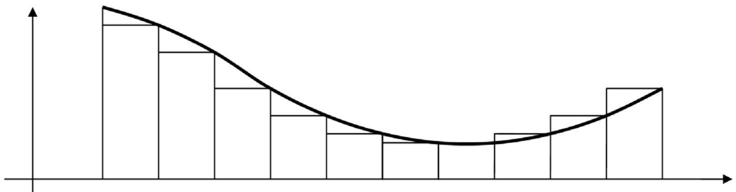
Limit Definition of the Definite Integral

Learning Outcomes: I can interpret the definite integral as the limit of a Riemann sum.

Using a Right Riemann Sum



1. Write an expression for the number of rectangles.

2. Write an expression for the width of each rectangle.

3. Write an expression for the height of each rectangle.

4. Write an expression for the area of each rectangle.

5. Write an expression for the sum of the areas of all the rectangles.

Definite Integral as the Limit of a Riemann Sum

If f is a continuous function defined on $[a, b]$, and if:

- $[a, b]$ is divided into n equal subintervals of width $\Delta x = \frac{b-a}{n}$
- $x_k = a + k\Delta x$ is the right endpoint of subinterval k.

then the definite integral of f from a to b is the number

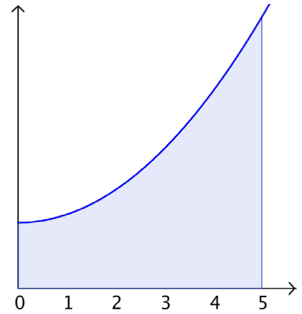
$$\int_a^b f(x)\,dx =$$

Let n be the number of subintervals. The exact area under the curve is given by the limit:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{5}{n}\right)\left(\left(\frac{5k}{n}\right)^2 + 2\right)$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\left(\frac{5k}{n}\right)^2 + 2\right) \Delta x$$

which is exactly equal to:



6. Write a limit that is equal to $\int_3^5 x^4 \, dx$.

7. Write a limit that is equal to $\int_2^5 (4 - 2x) \, dx$.

Summation Properies

1. $\sum_{k=1}^{n} c =$
2. $\sum_{k=1}^{n} ca_k =$
3. $\sum_{k=1}^{n} (a_k \pm b_k) =$

Summation Formulas

1. $\sum_{k=1}^{n} k =$
2. $\sum_{k=1}^{n} k^2 =$
3. $\sum_{k=1}^{n} k^3 =$

8. Use the definition of definite integral to evaluate $\int_3^5 (3x + 1)\,dx$.

   First, we write an expression for Δx.

   $$\Delta x =$$

   Second, we write an expression for x_k.

   $$x_k =$$

   Third, we write the limit definition of the definite integral.

   $$\int_3^5 (3x + 1)\,dx =$$

9. Use the definition of definite integral to evaluate $\int_0^4 (2x^2 + 3)\,dx$.

   First, we write an expression for Δx.

   $$\Delta x =$$

   Second, we write an expression for x_k.

   $$x_k =$$

   Third, we write the limit definition of the definite integral.

   $$\int_0^4 (2x^2 + 3)\,dx =$$

10. Use the definition of definite integral to evaluate $\int_2^5 (8x - x^2)\, dx$.

Limit Definition of the Definite Integral

1. Identify Δx and x_k, then express the following Riemann sums as definite integrals.

(a) $\lim_{n\to\infty} \sum_{k=1}^{n} 2\left(-2+\frac{3k}{n}\right)\frac{3}{n}$

(b) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{4k}{n}\right)\frac{4}{n}$

(c) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(1+\frac{2k}{n}\right)^3\frac{2}{n}$

(d) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(1+\left(\frac{2k}{n}\right)^3\right)\frac{2}{n}$

(e) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{5}{n}\sqrt{\frac{5k}{n}+3}$

(f) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{5}{n}\left(\sqrt{\frac{5k}{n}}-3\right)$

(g) $\lim_{n\to\infty} \sum_{k=1}^{n} \cos\left(\frac{\pi}{4}+\frac{\pi k}{2n}\right)\frac{\pi}{2n}$

(h) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{16k}{n^2}\right)\left(\ln\frac{4k}{n}\right)$

2. Use the limit definition of the definite integral to evaluate the following integrals.

(a) $\int_1^3 4x\,dx$

(b) $\int_0^4 (3x+4)\,dx$

(c) $\int_{-1}^2 (x-1)^2\,dx$

(d) $\int_1^3 (x^3+1)\,dx$