AP® Calculus AB **Lesson Plan**

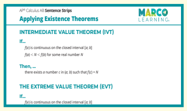
Applying Existence Theorems

Duration

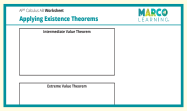
One 90-minute class period

Resources

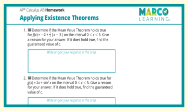
1. Sentence strips



2. Worksheet



3. Homework



SENTENCE STRIPS

For each group, print one copy of the sentence strip handout, cut into 24 strips, and then place into an envelope.

Objectives of Lesson

- Students will identify and check that the conditions of a theorem are satisfied prior to applying the conclusion of the theorem.
- Students will apply the following existence theorems to describe the behavior of a function over a given interval: the Intermediate Value Theorem, the Extreme Value Theorem, Rolle's Theorem, and the Mean Value Theorem.

College Board Objectives from the 2019–20 *CED*

- **Mathematical Practices—Practice 3:** Justification—Justify reasoning and solutions.
- **Mathematical Practices—Practice 3B:** Identify an appropriate mathematical definition, theorem, or test to apply.
- **Mathematical Practices—Practice 3C:** Confirm whether hypotheses or conditions of a selected definition, theorem, or test have been satisfied.
- **Mathematical Practices—Practice 3D:** Apply an appropriate mathematical definition, theorem, or test.
- **Mathematical Practices—Practice 3E:** Provide reasons or rationales for solutions and conclusions.
- **Mathematical Practices—Practice 4:** Communication and Notation—Use correct notation, language, and mathematical conventions to communicate results or solutions.
- **Mathematical Practices—Practice 4A:** Use precise mathematical language.
- **Mathematical Practices—Practice 4B:** Use appropriate units of measure.
- **Mathematical Practices—Practice 2:** Connect representations.
- **Mathematical Practices—Practice 2B:** Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.

How to Use This Lesson Plan

For the initial sorting activity, provide each group with an envelope containing the four theorems cut apart into "sentence strips." Task students with reconstructing the four theorems. To do so, they must determine which statements belong in which theorem, and within each theorem, which statements belong in the hypothesis and which belong in the conclusion. The purpose of this lesson is to help students analyze the parts of each of these theorems, then synthesize the information to reconstruct the theorems in order to gain a more in-depth understanding of what each part of each theorem means, when and how to apply these theorems to real situations, and to distinguish which theorems involve instantaneous rates of change and which do not.

Another broader goal of this lesson is for students to analyze the structure of a theorem from a logical approach. Students often apply theorems to problems without checking the necessary conditions and assumptions that must be satisfied in order to reach the desired conclusion. By breaking the hypotheses of these theorems into smaller parts and separating them from the conclusions, you can emphasize the importance of these conditions and distinguish the hypotheses from the conclusions.

In the second part of this lesson, students must solve problems in a real-world context, giving reasons for their answers. These problems require students to recognize when (this involves checking the conditions) and how to apply the different existence theorems they have learned without specifically being prompted to do so. Encourage students to expand upon and justify their thinking in order to conduct meaningful discussions within their small groups.

Instructions

1. Sorting Activity (10–15 minutes). Give each group an envelope containing the 24 sentence strips. Tell them their task is to sort the strips into four theorems. Circulate around the room, and once a group has reconstructed the theorems, check that they are correct. If not, ask a guided question to help them identify and correct their error. Then, on the front page of the student worksheet, students will copy the theorems in the appropriate boxes.

2. Follow-Up Questions (10 minutes). Debrief with a whole-group discussion
   a. What do all four theorems have in common?
   b. Which theorems require differentiability? Why don't the others?
   c. How did you decide whether a strip was an "if" or a "then"?
   d. What would happen if you removed one of the "if" strips? Why?
3. Application (60 minutes). Have students continue working together to complete the set of problems on the student worksheet. Allow them to use calculators. Circulate around the room and facilitate small-group discussions as needed.

Applying Existence Theorems

INTERMEDIATE VALUE THEOREM (IVT)

If...

$f(x)$ is continuous on the closed interval $[a; b]$

$f(a) < N < f(b)$ for some real number N

Then, ...

there exists *a* number c in $(a; b)$ such that $f(c) = N$

THE EXTREME VALUE THEOREM (EVT)

If...

$f(x)$ is continuous on the closed interval $[a; b]$

Then, ...

the absolute maximum and absolute minimum values of f are guaranteed to occur at $x = a$, $x = b$, or $x = c$, for some c in $(a; b)$ such that $f'(c) = 0$ or undened

ROLLE'S THEOREM

If...

$f(x)$ is continuous on the closed interval $[a; b]$

$f(x)$ is dierentiable on $(a; b)$

$f(a) = f(b)$

Then, ...

there is at least one value c on $(a; b)$ at which $f'(c) = 0$

MEAN VALUE THEOREM (MVT)

If...

$f(x)$ is continuous on the closed interval $[a; b]$

$f(x)$ is dierentiable on $(a; b)$

Then, ...

there is at least one value c on $(a; b)$ at which $f'(c) = \frac{f(b) - f(a)}{b - a}$

Applying Existence Theorems

Intermediate Value Theorem

Extreme Value Theorem

Rolle's Theorem

Mean Value Theorem

1. The rate at which customer service calls are handled, in calls per hour, is given by a differentiable function *C(t)*. The table below shows the rate as measured every 3 hours for a 24-hour period.

t (hrs)	0	3	6	9	12	15	18	21	24
$C(t)$ (calls/hr)	19.6	20.4	20.8	21.2	21.4	21.3	20.7	20.2	19.6

(a) Estimate the value of $C'(5)$, indicating correct units of measure. Explain what this means about *C(t)*.

Write or type your response in this area.

(b) Using correct units of measure, find the average rate of change of *C(t)* from $t = 3$ to $t = 18$.

Write or type your response in this area.

(c) Is there some time t, $0 < t < 24$, such that $C'(t) = 0$? Justify your answer.

Write or type your response in this area.

2. The total cost $C(x)$, measured in dollars, of fidget spinners is approximated by the function

$$C(x) = 100\left(\frac{1}{x} + \frac{x}{x+3}\right)$$

where x is the order size in number of fidget spinners.

(a) Is there guaranteed a value of x on the interval $0 \leq x \leq 3$ such that the average rate of change of the total cost is equal to $C'(x)$? Give a reason for your answer.

Write or type your response in this area.

(b) Is there a value of x on the interval $3 \leq x \leq 6$ such that $C'(t) = 0$? Give a reason for your answer and if such a value of x exists, find the value.

Write or type your response in this area.

(c) For $3 \leq x \leq 9$, what is the greatest total cost of fidget spinners?

Write or type your response in this area.

3. A restaurant introduces a new smoothie for which the number of smoothies sold, S, is modeled by the function

$$S(t) = 300\,\left(5 - \frac{9}{t+2}\right)$$

where t is the time in months.

(a) Find the value of $S'(2{:}5)$. Using correct units, explain what this value represents in the context of this problem.

Write or type your response in this area.

(b) Find the average rate of change of smoothies sold over the first 12 months. Indicate correct units of measure, and explain what this value represents in the context of this problem.

Write or type your response in this area.

(c) Is it possible that a value of c for 0 c 12 exists such that $S'(c)$ is equal to the average rate of change? Give a reason for your answer and if such a value of c exists, find the value.

Write or type your response in this area.

4. The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table shows values of the functions and their first derivatives at selected values of x. The function h is given by the equation $h(x) = f(g(x)) - 6$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	-4	4	2
4	-1	3	6	7

(a) Find the equation of the tangent line drawn to the graph of h when $x = 3$.

Write or type your response in this area.

(b) Find the average rate of change of h for the interval $1 < x < 3$.

Write or type your response in this area.

(c) Explain why there must be a value of r for $1 < r < 3$ such that $h(r) = -2$.

Write or type your response in this area.

(d) Explain why there is a value of c for $1 < c < 3$ such that $h'(c) = -5$.

Write or type your response in this area.

Applying Existence Theorems

1. Determine if the Mean Value Theorem holds true for $f(x) = -2 + \frac{1}{2}\,|x - 3|$ on the interval $0 < c < 5$. Give a reason for your answer. If it does hold true, find the guaranteed value of c.

Write or type your response in this area.

2. Determine if the Mean Value Theorem holds true for $g(x) = 2x + \sin^2 x$ on the interval $0 < c < 5$. Give a reason for your answer. If it does hold true, find the guaranteed value of c.

Write or type your response in this area.

3. For $t \geq 0$, the temperature of a cup of coffee in degrees Fahrenheit t minutes after it is poured is modeled by the function $F(t) = 68 + 93\,(0{:}91)^t$. Use your calculator to find the value of $F'(4)$. Using correct units of measure, explain what this value means in the context of the problem.

Write or type your response in this area.

4. Administrators at a hospital believe that the number of beds in use is given by the function

$$B(t) = 20 \sin\left(\tfrac{t}{10}\right) + 50$$

where t is measured in days.

(a) Use your calculator to find the value of $B'(7)$. Using correct units of measure, explain what this value means in the context of the problem.

Write or type your response in this area.

(b) For $12 \leq t \leq 20$, what is the maximum number of beds in use?

Write or type your response in this area.

Use the table given below which represents values of a differentiable function g on the interval $0 \leq x \leq 6$. Be sure to completely justify your reasoning, citing appropriate theorems where applicable.

x	0	2	3	4	6
$g(x)$	-3	1	5	2	1

5. Estimate the value of $g'(2.5)$.

Write or type your response in this area.

6. If one exists, on what interval is there guaranteed to be a value of c such that $g(c) = -1$? Justify your reasoning.

Write or type your response in this area.

7. If one exists, on what interval is there guaranteed to be a value of c such that $g'(c) = 0$? Justify your reasoning.

Write or type your response in this area.

8. If one exists, on what interval is there guaranteed to be a value of c such that $g'(c) = 4$? Justify your reasoning.

Write or type your response in this area.